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Graetz Problem for the Conjugated Conduction-Film Condensation Process

O. Bautista,* F. Méndez,† and C. Treviño‡
Universidad Nacional Autonoma de Mexico, 04510 Mexico, D.F., Mexico

We study the condensation process of a saturated vapor in contact with the external surfaces of two vertical thin plates. The condensation is caused by a forced cooling flow between the internal surfaces of the plates. The longitudinal heat conduction effects in the plate are considered. By use of perturbation techniques, the governing equations are reduced to a system of integro-differential equations with five nondimensional parameters: the Prandtl number of the condensed fluid Pr_c, the Jakob number Ja, the nondimensional plate thermal conductivity α, the aspect ratio of the plate ε, and the ratio of the thermal conductance of the condensed layer to the thermal conductance of the forced cooling flow β. The condensed-layer thickness and the temperature of the plates are estimated in the asymptotic limit $Ja \rightarrow 0$ and $\alpha/\varepsilon^2 \gg 1$ (thermally thin wall regime).

Nomenclature

c = specific heat of the forced fluid
c_c = specific heat of the condensed fluid
f_c = nondimensional stream function
g = magnitude of the acceleration due to gravity
h = thickness of the plate
h_{fg} = latent heat of condensation
L = length of the plate
m' = mass flow rate of condensed fluid
Pr = Prandtl number of the cooling fluid, $\nu\rho c/\lambda$
Pr_c = Prandtl number of the condensed fluid, $\nu_c\rho_c c_c/\lambda_c$
Re = Reynolds number of the cooling fluid, $2H\bar{U}/\nu$
T = temperature in physical units for the fluids and the plates
T_s = temperature of the saturated vapor
T_∞ = freestream temperature of the forced cooling fluid flow
u, v = nondimensional longitudinal and transversal components of the speed
$\bar{u}$, $\bar{v}$ = longitudinal and transversal components of the speed in physical units
x, y = Cartesian coordinates
Δ = normalized thickness of the condensed fluid
δ = thickness of the forced cooling fluid
δ_c = thickness of the condensed layer
δ_{cL} = thickness of the condensed layer at $\chi = 1$
ε = aspect ratio of the plate
η_c = nondimensional transversal coordinate for the condensed fluid
θ = nondimensional temperature of the forced cooling fluid
θ_c = nondimensional temperature of the condensed fluid
θ_w = nondimensional temperature of the plate
λ = thermal conductivity of the forced cooling fluid
λ_c = thermal conductivity of the condensed fluid
λ_w = plate thermal conductivity
μ_c = dynamic viscosity of the condensed fluid
ν = kinematic coefficient of viscosity of the forced cooling fluid
ν_c = kinematic coefficient of viscosity of the condensed fluid
ρ = density of the cooling fluid
ρ_c = density of the condensed fluid

Subscripts

c = condensed fluid
e = $\alpha = 0$
L = conditions at the lower end of the wall
l = conditions at the upper end of the wall
w = conditions at the wall

I. Introduction

THE heat transfer analysis of conjugated laminar film condensation on surfaces is an important field in engineering for the thermal design of heat exchangers. Since the pioneering paper of Nusselt,[1] a standard practice in the conventional heat transfer studies has been to assume uniform thermal boundary conditions for the solid body that interacts with the fluid flow. Several works have appeared in the specialized literature, and the state of the art for isothermal surfaces is demonstrated by Rose[2] and recently by Tanasawa.[3] However, this is an idealized scheme. Basically, the convective heat transfer coefficient on an arbitrary surface is established by a coupled thermal interaction between the surface and the surrounding flow. Therefore, the flow and temperature fields have a considerable influence on the convective heat transfer coefficient that modifies the temperature distribution at the solid surface. The simplifications introduced by isothermal boundary conditions should be reexamined to have more realistic heat transfer models. Along these lines, Patankar and Sparrow[4] numerically solved the laminar film condensation process on a cooled nonisothermal vertical fin or cylinder, using a similarity analysis. The condensation process has been coupled to the heat conduction within the fin. They concluded that the calculated heat transfer is lower than that predicted by using an isothermal fin model. Wilkins[5] obtained an explicit analytical solution for this problem. Thus, studies of condensation on extended surfaces form a class by themselves, and an estimation of the surface area requirements of condensers using classical Nusselt analysis is not appropriate. To extend the analyses to nonisothermal cases, Sarma et al.[6] studied the condensation process on a vertical fin of variable thickness. Brouwers[7] developed an analytical study for the condensation of a pure saturated vapor on a cooled channel plate, including the thermal interaction between the cooling liquid, the condensate, and the vapor. The governing equations were solved numerically, analyzing different cases for co-, counter-, and cross-current condensation processes. Méndez and Treviño[8] solved the problem of film condensation on a thin vertical surface caused by a forced cooling fluid, showing that the local heat forced convection through the lateral surface of the plate, strongly affected by the axial heat conduction, governs the spatial evolution of the plate temperature and the condensed-layer thickness. Similar results

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*Postgraduate Student, Mechanical Engineering, Facultad de Ingeniería.
†Professor, Mechanical Engineering, Facultad de Ingeniería.
‡Professor, Mechanical Engineering, Facultad de Ciencias.

for other cases were reported by Treviño et al.[9] and Méndez and Treviño.[10]

In this paper, we use perturbation techniques and the classical boundary-layer theory for the analysis of cooling forced and condensed flows. We study the conjugated film-condensation process generated by a Graetz cooling flow on the external surfaces of two flat vertical plates. We conduct an asymptotic analysis for large and small values of the thermal conductance parameter α and compare the approximate analytical solutions with the numerical results.

II. Formulation and Order-of-Magnitude Analysis

The physical model under study is shown in Fig. 1. Two thin vertical heat conducting plates, separated by a distance H, length L, and thickness h, are placed in a stagnant atmosphere filled with saturated vapor with uniform temperature T_s. For simplicity, both ends of the plates are connected with adiabatic surfaces. A Hagen–Poiseuille convective flow with mean velocity $\bar{U}$ and uniform temperature $T_\infty < T_s$ is imposed between the plates, generating a heat flux from the saturated vapor and creating thin condensed films at the external surfaces of the plates. The density of the condensed fluid ρ_c is assumed to be constant and much larger than the vapor density ρ. The Cartesian coordinates system is placed at the symmetry vertical plane, as shown in Fig. 1. The y axis points out in the direction normal to the plates and the x axis downward in the longitudinal direction. Because of the geometrical and physical symmetries, we formulate the governing equations only for positive values of the y coordinate. We introduce an order-of-magnitude analysis to obtain the nondimensional parameters and the global relationships between them.

Using the classical Nusselt[1] assumptions for the condensed fluid, the characteristic velocity, the condensed mass flow and the mass production rate are of order

$$u_c \sim \frac{g\delta_c^2(x)}{\nu_c}, \qquad m' \sim \frac{\rho_c g\delta_c^3(x)}{\nu_c}, \qquad \frac{\mathrm{d}m'}{\mathrm{d}x} \sim \frac{\lambda_c \Delta T_c}{\delta_c(x)h_{fg}} \tag{1}$$

respectively, where ΔT_c is the characteristic temperature difference in the condensed fluid. From these relationships, the global thickness of the condensed layer related to the length of the plate is given by

$$\frac{\delta_{cL}}{L} \sim \left(\frac{Ja}{\gamma}\frac{\Delta T_c}{\Delta T}\right)^{\frac{1}{4}} \quad \text{with} \quad Ja = \frac{c_c \Delta T}{h_{fg}Pr_c}, \qquad \gamma = \frac{gL^3}{\nu_c^2} \tag{2}$$

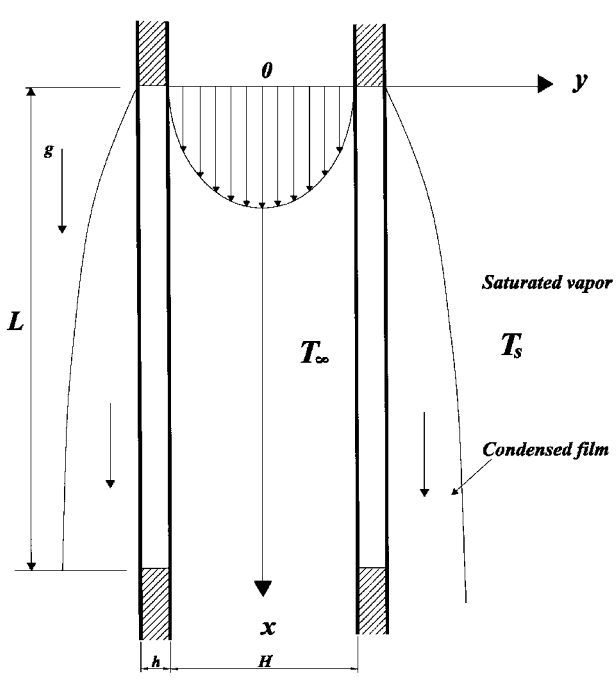

Fig. 1 Schematic diagram of the physical model.

where the Jakob number Ja is the ratio of the sensible thermal energy absorbed by the liquid to the latent heat of the liquid during the condensation process and ΔT is the global temperature difference $\Delta T = T_s - T_\infty$. In general, the Jakob number and the parameter γ are very small and large compared with unity, respectively,[11] and the limit $Ja/\gamma \rightarrow 0$ is fully justified. The Graetz cooling flow can be analyzed by introducing a thermal boundary layer for high values of the Péclet number, $Pe = RePr$. It can be easily shown that the corresponding thickness of the thermal boundary layer is

$$\delta \sim [H^2/L^2 Pe]^{\frac{1}{3}} L \tag{3}$$

The global temperature drop ΔT from the condensed fluid to the cooling flow is related to each temperature drop of the system as $\Delta T \sim \Delta T_c + \Delta T_w + \Delta T_\infty$, where ΔT_w and ΔT_∞ are the characteristic temperature differences across the plate and the Graetz flow, respectively. By equating the heat fluxes in both surfaces of the plate and using relationships (1) and (3), we obtain

$$\frac{\Delta T_c}{\Delta T} \sim \left(\frac{\alpha}{\varepsilon^2}\right)^{\frac{4}{3}} \left(\frac{\Delta T_w}{\Delta T}\right)^{\frac{4}{3}}, \qquad \frac{\Delta T_\infty}{\Delta T} \sim \frac{\alpha}{\varepsilon^2 \bar{\beta}}\frac{\Delta T_w}{\Delta T} \tag{4}$$

and introducing the temperature drop ΔT together with Eq. (4), we get

$$\frac{\Delta T_w}{\Delta T}\left(1 + \frac{\alpha}{\varepsilon^2 \bar{\beta}}\right) + \left(\frac{\alpha}{\varepsilon^2}\right)^{\frac{4}{3}} \left(\frac{\Delta T_w}{\Delta T}\right)^{\frac{4}{3}} \sim 1 \tag{5}$$

The relevant parameters $\bar{\beta}$, α, and ε are defined as

$$\bar{\beta} = \frac{\lambda}{\lambda_c}\frac{(Ja/\gamma)^{\frac{1}{4}}}{\left[\frac{1}{3}(H^2/L^2 Pe)\right]^{\frac{1}{3}}}, \qquad \alpha = \frac{\lambda_w}{\lambda_c}\frac{h}{L}\left(\frac{Ja}{\gamma}\right)^{\frac{1}{4}}, \qquad \varepsilon = \frac{h}{L} \tag{6}$$

where $\bar{\beta}$ is the ratio of the thermal resistance of the condensed fluid to the thermal resistance of the cooling flow. The parameter α relates the heat conducted by the plate to the heat convected from the condensed vapor fluid. Here ε is the aspect ratio of the plate $\varepsilon = h/L$, which is assumed to be very small compared with unity. From relationships (4) and (5), it can be shown that, for values of $\bar{\beta}$ of the order unity and $\alpha/\varepsilon^2 \gg 1$,

$$\frac{\Delta T_w}{\Delta T} \sim \frac{\varepsilon^2}{\alpha} \ll 1, \qquad \frac{\Delta T_c}{\Delta T} \sim 1, \qquad \frac{\Delta T_\infty}{\Delta T} \sim 1 \tag{7}$$

Thus, for values of α/ε^2 very large compared with unity, the transverse temperature variations at the plate are very small compared with the overall temperature difference. This limit corresponds to the thermally thin wall regime, and because of the practical applications, we develop analytical and numerical solutions for this regime in the following sections.

III. Governing Equations

The nondimensional governing equations for the flat plate and the condensed fluid are given next, together with the corresponding initial and boundary conditions. For this conjugated heat transfer problem, we add the thermal compatibility conditions. Finally, the integral solution obtained using the Duhamel's theorem for the Graetz cooling flow is also presented. By introducing the nondimensional variables

$$\theta_w = \frac{T_s - T_w}{T_s - T_\infty}, \qquad \chi = \frac{x}{L}, \qquad z = \frac{y - H/2}{h}$$

$$u = \frac{\bar{u}}{\sqrt{gLJa}} = \Delta^2 \frac{\partial f_c}{\partial \eta_c}, \qquad v = \frac{\bar{v}\gamma^{\frac{1}{4}}}{Ja^{\frac{3}{4}}\sqrt{gL}} = -\Delta \frac{\partial(\Delta^2 f_c)}{\partial \chi}$$

$$\theta_c = \frac{T_s - T_c}{T_s - T_\infty}, \qquad \Delta = \frac{\delta_c(x)}{L(Ja/\gamma)^{\frac{1}{4}}}, \qquad \eta_c = \frac{y - (H/2 + h)}{\delta_c(x)} \tag{8}$$

the nondimensional governing equations transform as follows.

Plate:

$$\frac{\partial^2 \theta_w}{\partial \chi^2} + \frac{\partial^2 \theta_w}{\partial z^2} = 0 \tag{9}$$

Condensed flow:

$$\frac{\partial^3 f_c}{\partial \eta_c^3} + 1 = Ja\Delta^4 \left\{ \frac{\partial f_c}{\partial \eta_c}\frac{\partial^2 f_c}{\partial \chi \partial \eta_c} - \frac{\partial f_c}{\partial \chi}\frac{\partial f_c}{\partial \eta_c} + \frac{2}{\Delta}\frac{\mathrm{d}\Delta}{\mathrm{d}\chi}\left[\left(\frac{\partial f_c}{\partial \eta_c}\right)^2 - f_c \frac{\partial^2 f_c}{\partial \eta_c^2}\right]\right\} \tag{10}$$

$$\frac{\partial^2 \theta_c}{\partial \eta_c^2} = JaPr\Delta^4 \left\{ \frac{\partial f_c}{\partial \eta_c}\frac{\partial \theta_c}{\partial \chi} - \frac{\partial f_c}{\partial \chi}\frac{\partial \theta_c}{\partial \eta_c} - \frac{2}{\Delta}\frac{\mathrm{d}\Delta}{\mathrm{d}\chi} f_c \frac{\partial \theta_c}{\partial \eta_c}\right\} \tag{11}$$

The momentum and energy equations for the condensed fluid were derived using the boundary-layer approximation. The adiabatic boundary conditions for the plate are given by

$$\left.\frac{\partial \theta_w}{\partial \chi}\right|_{\chi=0,1} = 0 \tag{12}$$

whereas the boundary conditions associated with the condensed fluid are

$$f_c(\chi, 0) = \left.\frac{\partial f_c}{\partial \eta_c}\right|_{\eta_c=0} = 0, \qquad \theta_c(\chi, 1) = \left.\frac{\partial^2 f_c}{\partial \eta_c^2}\right|_{\eta_c=1} = 0 \tag{13}$$

The last condition in Eq. (13) arises from the balance of tangential shear stresses at the interface.[2] Furthermore, we also need the compatibility conditions at both surfaces of the plate. They are the continuity in the temperatures and heat fluxes and may be written as follows.

External surface:

$$\theta_w(\chi, 1) - \theta_c(\chi, 0) = 0, \qquad \left.\frac{\partial \theta_w}{\partial z}\right|_{z=1} = \frac{\varepsilon^2}{\alpha}\frac{1}{\Delta}\left.\frac{\partial \theta_c}{\partial \eta_c}\right|_{\eta_c=0} \tag{14}$$

Internal surface:

$$\theta_w(\chi, 0) - \theta(\chi, 0) = 0, \qquad \left.\frac{\partial \theta_w}{\partial z}\right|_{z=0} = -\left(\frac{4}{3}\right)^{\frac{1}{3}} \frac{\bar{\beta}\varepsilon^2}{\alpha}\frac{1}{\chi^{\frac{1}{3}}}\left.\frac{\partial \theta}{\partial \zeta}\right|_{\zeta=0} \tag{15}$$

The nondimensional variables θ and ζ for the cooling flow are defined by

$$\theta = \frac{T_s - T}{T_s - T_\infty}, \qquad \zeta = \left(\frac{HPe}{2L}\right)^{\frac{1}{3}} \frac{(1 - 2y/H)}{\chi^{\frac{1}{3}}} \tag{16}$$

The normalized nondimensional thickness of the condensed film, $\Delta(\chi)$, is unknown and must be obtained from the analysis. Thus, the energy balance at the condensed-vapor interface provides the spatial evolution of Δ as

$$4\Delta \frac{\mathrm{d}\left[\Delta^3 f_c(\chi, 1)\right]}{\mathrm{d}\chi} = -\left.\frac{\partial \theta_c}{\partial \eta_c}\right|_{\eta_c=1} \quad \text{with} \quad \Delta(\chi = 0) = 0 \tag{17}$$

Cooling flow:

$$\frac{\partial^2 \theta}{\partial \zeta^2} + \frac{1}{4}\zeta^2 \frac{\partial \theta}{\partial \zeta} = \frac{3}{4}\zeta\chi\frac{\partial \theta}{\partial \chi} \tag{18}$$

together with boundary conditions

$$\theta(\chi, \infty) - 1 = \left.\frac{\partial \theta}{\partial \zeta}\right|_{\zeta\to\infty} = 0 \tag{19}$$

Here the Péclet number is assumed to be very large compared with unity. This linear energy equation can be easily integrated by using the Duhamel's theorem. (The details are omitted for simplicity.) For this case, the nondimensional heat flux at the inner surface of the plate can be written as[12]

$$\left.\frac{\partial \theta}{\partial \zeta}\right|_{\zeta=0} = \frac{1}{48\Gamma(\frac{1}{3})\chi^{\frac{1}{3}}}\left[1 - \theta_{wl} - \int_0^\chi K(\chi, \chi')\frac{\mathrm{d}\theta_w'}{\mathrm{d}\chi'}\,\mathrm{d}\chi'\right] \tag{20}$$

where the kernel K for laminar Graetz flow is given by $K(\chi, \chi') = (1 - \chi'/\chi)^{-1/3}$ and θ_{wl} is the value of the nondimensional temperature at the upward end of the plate.

IV. Thermally Thin Wall Limit ($\alpha/\varepsilon^2 \gg 1$)

In this regime, the nondimensional temperature of the plate depends only on the χ coordinate in a first approximation, as predicted in relationship (7). Therefore, Eq. (9) can be integrated along the transverse coordinate and, after applying the compatibility conditions (14) and (15) at both vertical surfaces of the plate, together with Eq. (20), we obtain

$$\alpha\frac{\mathrm{d}^2\theta_w}{\mathrm{d}\chi^2} + \frac{1}{\Delta}\left.\frac{\partial \theta_c}{\partial \eta_c}\right|_{\eta_c=0} = \frac{\beta}{\chi^{\frac{1}{3}}}\left[\theta_{wl} - 1 + \int_{\theta_{wl}}^{\theta_w} K(\chi, \chi')\,\mathrm{d}\theta_w'\right] \tag{21}$$

The second term on the left-hand side of Eq. (21) denotes the heat transferred from the condensed fluid, and the term on the right-hand side is the heat transferred to the cooling flow. The boundary conditions at the plate ends transform to

$$\left.\frac{\mathrm{d}\theta_w}{\mathrm{d}\chi}\right|_{\chi=0,1} = 0 \tag{22}$$

with the nondimensional parameter β defined by

$$\beta = \frac{4^{\frac{1}{3}}\bar{\beta}}{48\Gamma(\frac{1}{3})3^{\frac{1}{3}}} \doteq 0.00856\bar{\beta} \tag{23}$$

For values of $\beta \gg 1$, the nondimensional plate temperature is practically 1, and the laminar film condensation process corresponds to the Nusselt's limit to a first approximation. A uniform temperature of the plate also is obtained for large values of α, where the temperature profiles depend on the values of β. The limiting case of $\beta = 0$ represents a simple conjugated heat transfer process between the plate and the cooling flow, without any condensation phenomena. The system of Eqs. (10), (11), (17), and (21), together with the given initial and boundary conditions, contains three differential equations and one integro-differential equation, respectively, for the unknowns $f_c(\chi, \eta_c)$, $\theta_c(\chi, \eta_c)$, $\Delta(\chi)$, and $\theta_w(\chi)$ with five different nondimensional parameters, Ja, Pr_c, α, β, and ε. In the following sections we analyze the limit characterized by small Jakob number,[11] $Ja \to 0$, with the Prandtl number of the condensed phase of order unity. The limits of large and small values of α and β of order unity for the thermally thin wall regime ($\alpha/\varepsilon^2 \gg 1$) are considered.

A. Asymptotic Solution for $Ja \to 0$

In this limit and assuming Pr_c to be of order unity, the solution of the governing equations for the condensed phase equations (10) and (11) with Eq. (13) are given by

$$\theta = \theta_w(\chi)(1 - \eta_c), \qquad f_c(\eta_c) = \tfrac{1}{2}\eta_c^2(1 - \eta_c/3) \tag{24}$$

The appropriate or reduced Nusselt number for this problem, Nu_c^*, is

$$Nu_c^* = \frac{qL}{\lambda_c(T_s - T_\infty)}\left(\frac{Ja}{\gamma}\right)^{\frac{1}{4}} = -\frac{1}{\Delta}\left.\frac{\partial \theta_c}{\partial \eta_c}\right|_{\eta_c=0} = \frac{\theta_w}{\Delta} \tag{25}$$

Thus, the nondimensional energy balance equation (17) transforms to

$$\frac{d\Delta^4}{d\chi} = \theta_w \tag{26}$$

and Eq. (21) for the solid can now be rewritten as

$$\alpha\frac{d^2\theta_w}{d\chi^2} - \frac{\theta_w}{\Delta} = \frac{\beta}{\chi^{\frac{1}{3}}}\left[\theta_{wl} - 1 + \int_{\theta_{wl}}^{\theta_w} K(\chi,\chi')\,d\theta_w'\right] \tag{27}$$

In this limit, the solution depends on two free parameters: α and β. The nonlinear system of Eqs. (26) and (27) must be solved with the initial condition $\Delta(0) = 0$ and the adiabatic conditions at the ends of the plate given by Eqs. (22). We use numerical techniques[8] to solve these equations, and we have explored asymptotic solutions for large and small values of the parameter α.

1. Solution for $\alpha \gg 1$

In this limit the solution is regular and the nondimensional temperature of the plate θ_w changes very little (of the order α^{-1}) in the longitudinal direction. We assume that the nondimensional temperature of the plate, as well as the nondimensional condensed-layer thickness, can be expanded as follows:

$$\theta_w(\chi) = \theta_0(\chi) + \sum_{j=1}^{\infty}\alpha^{-j}\theta_j(\chi)$$

$$\Delta(\chi) = \Delta_0(\chi) + \sum_{j=1}^{\infty}\alpha^{-j}\Delta_j(\chi) \tag{28}$$

Introducing these relationships into Eqs. (26) and (27), we obtain, after collecting terms of the same power of α, the following sets of equations:

$$\frac{d^2\theta_0}{d\chi^2} = 0, \qquad \frac{d\Delta_0^4}{d\chi} = \theta_0 \tag{29}$$

$$\frac{d^2\theta_1}{d\chi^2} = \frac{\theta_0}{\Delta_0} + \frac{\beta}{\chi^{\frac{1}{3}}}(\theta_0 - 1), \qquad \frac{4\,d(\Delta_0^3\Delta_1)}{d\chi} = \theta_1 \tag{30}$$

$$\frac{d^2\theta_2}{d\chi^2} = \frac{\theta_0}{\Delta_0}\left(\frac{\theta_1}{\theta_0} - \frac{\Delta_1}{\Delta_0}\right) + \frac{\beta}{\chi^{\frac{1}{3}}}\left[\theta_{1l} + \int_{\theta_{1l}}^{\theta_1} K(\chi,\chi')\,d\theta_1'\right] \tag{31}$$

$$\frac{d(4\Delta_0^3\Delta_2 + 6\Delta_0^2\Delta_1^2)}{d\chi} = \theta_2 \tag{32}$$

and so forth, with the following initial and boundary conditions:

$$\Delta_i(0) = 0, \qquad \left.\frac{d\theta_i}{d\chi}\right|_{\chi=0,1} = 0 \qquad \text{for all} \qquad i \geq 0 \tag{33}$$

Integration of Eqs. (29) with the corresponding initial and boundary conditions (33) gives $\theta_0 = C_0$ and $\Delta_0 = C_0^{1/4}\chi^{1/4}$, with a constant $C_0(\beta)$ to be obtained in the next order. We can integrate Eq. (30), and on applying the adiabatic boundaries at both ends of the plate, we obtain the leading order of the nondimensional plate temperature as a function of the parameter β in the form

$$\beta = \frac{8C_0^{\frac{3}{4}}}{9(1 - C_0)} \tag{34}$$

To the leading order of Eq. (3), the nondimensional thickness of the condensate at the lower end of the plate that is a measure of the condensation efficiency, after employing the limits of large and small values of β compared with unity, is given by

$$\Delta_0(1) \sim \left(\tfrac{9}{8}\right)^{\frac{1}{3}}\beta^{\frac{1}{3}} - \left(\tfrac{1}{3}\right)\left(\tfrac{9}{8}\right)^{\frac{5}{3}}\beta^{\frac{5}{3}} \qquad \text{for} \qquad \beta \to 0 \tag{35}$$

$$\Delta_0(1) \sim 1 - (2/9\beta) + (8/27\beta^2) \qquad \text{for} \qquad \beta \to \infty \tag{36}$$

Introducing the solutions for θ_0 and Δ_0 into Eq. (30) and integrating this equation twice, we obtain, after considering the appropriate initial and boundary conditions,

$$\theta_1 = \tfrac{3}{2}\beta(1 - C_0)\left[C_1(\beta) + \left(\tfrac{4}{7}\right)\chi^{\frac{7}{4}} - \tfrac{3}{5}\chi^{\frac{5}{3}}\right] \tag{37}$$

where C_1 is an integration constant related to the temperature of the plate at the top end and must be determined by solving the next-higher-order equation. Thus, the first-order correction to the nondimensional condensed-layer thickness gives

$$\Delta_1 = \tfrac{1}{3}\left[C_1(\beta)\chi^{\frac{1}{4}} - \tfrac{9}{40}\chi^{\frac{23}{12}} + \tfrac{16}{77}\chi^2\right] \tag{38}$$

Integration of Eq. (31) gives the value of C_1, after applying the adiabatic boundary conditions at both ends, as

$$C_1(\beta) = \frac{0.0172 + 0.1355\beta C_0^{\frac{1}{4}}}{1 + 1.7142\beta C_0^{\frac{1}{4}}} \tag{39}$$

Therefore, up to first order, the condensed-layer thickness is given by

$$\Delta = \chi^{\frac{1}{4}}\left[C_0^{\frac{1}{4}} + (1/3\alpha)\left(C_1 - \tfrac{9}{40}\chi^{\frac{5}{3}} + \tfrac{16}{77}\chi^{\frac{7}{4}}\right)\right] + \mathcal{O}(\alpha^{-2}) \tag{40}$$

The nondimensional plate temperature is then

$$\theta_w = C_0 + \tfrac{3}{2}[\beta(1 - C_0)/\alpha]\left(C_1 - \tfrac{3}{5}\chi^{\frac{5}{3}} + \tfrac{4}{7}\chi^{\frac{7}{4}}\right) + \mathcal{O}(\alpha^{-2}) \tag{41}$$

and the reduced Nusselt number now takes the form

$$Nu_c^* = \frac{C_0^{\frac{3}{4}}}{\chi^{\frac{1}{4}}}\left[1 + \frac{1}{C_0^{\frac{1}{4}}\alpha}\left(C_1 - \frac{115}{200}\chi^{\frac{5}{3}} + \frac{1120}{1617}\chi^{\frac{7}{4}}\right)\right] + \mathcal{O}(\alpha^{-2}) \tag{42}$$

The leading term on the right-hand side of the preceding equations represents the classical Nusselt solution[1] for an isothermal plate.

2. Solution for $\alpha \to 0$

For small values of α compared with unity, the longitudinal heat conduction term in Eq. (27) can be neglected to a first approximation. For values of $\alpha \to 0$ but large compared with ε^2, the thermally thin wall approximation is still valid. This case represents a singular limit due to the existence of two longitudinal heat conducting layers at both ends to satisfy the adiabatic boundary conditions. The structure of these thermal boundary layers is not presented because they have only a local thermal influence. Outside of these inner zones, longitudinal heat conduction through the plate is negligible, reducing the governing equations up to the leading order to the following set:

$$\frac{\theta_{we}}{\Delta_e} = -\frac{\beta}{\chi^{\frac{1}{3}}}\left[\int_1^{\theta_{we}} K(\chi,\chi')\,d\theta_{we}'\right], \qquad \frac{d\Delta_e^4}{d\chi} = \theta_{we} \tag{43}$$

To solve the preceding system, we need the initial conditions to be obtained from the matching with the inner heat conducting zone. For simplicity, these matching conditions are omitted. Equations (43) were solved numerically, and the results are shown in Figs. 2 and 3,

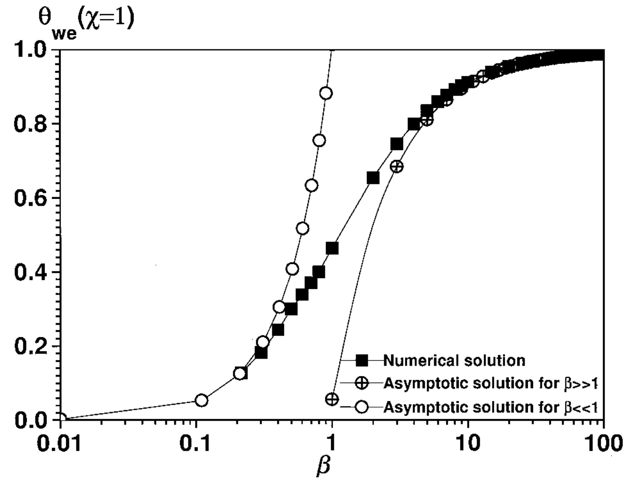

Fig. 2 Nondimensional temperature at the lower end of the plate ($\chi = 1$) as a function of β for $\alpha = 0$; also, asymptotic solution for large (first-order) and small values of β.

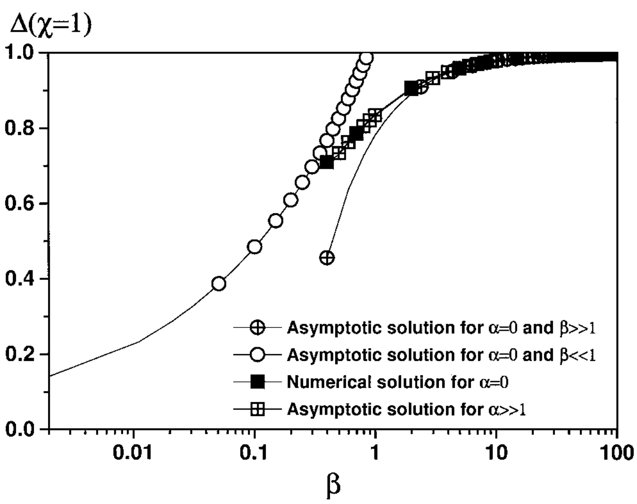

Fig. 3 Nondimensional condensed-layer thickness at the lower end of the plate ($\chi = 1$) as a function of β for $\alpha = 0$; also, asymptotic solutions for large (first-order) and small values of β and the leading-term solution for $\alpha \to \infty$.

using the earlier developed scheme.[10] To complete this section, we present asymptotic solutions of Eqs. (43) for large and small values of the parameter β. Equations (43) can be transformed to a parameter-free system given by

$$\frac{\theta_{we}}{\psi^{\frac{1}{4}}} = -\frac{1}{\sigma^{\frac{1}{3}}}\int_0^{\sigma} \frac{\mathrm{d}\theta'_{we}}{\mathrm{d}\sigma'} K(\sigma, \sigma')\,\mathrm{d}\sigma', \qquad \frac{\mathrm{d}\psi}{\mathrm{d}\sigma} = \theta_{we} \tag{44}$$

where we introduce the following variables:

$$\psi = \frac{\Delta_e^4}{\beta^{12}}, \qquad \sigma = \frac{\chi}{\beta^{12}} \tag{45}$$

It can be easily verified that the asymptotic solutions for the nondimensional plate temperature and thickness of the condensate after applying the initial conditions for high values of β are

$$\theta_{we} \sim 1 - \left[\frac{12\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{12})\Gamma(\frac{2}{3})}\right]\frac{\chi^{\frac{1}{12}}}{\beta} + \left\{\frac{720\Gamma(\frac{3}{4})\Gamma(\frac{5}{6})}{13\left[\Gamma(\frac{2}{3})\right]^2\Gamma(\frac{1}{12})\Gamma(\frac{1}{6})}\right\}\frac{\chi^{\frac{1}{6}}}{\beta^2} + \mathcal{O}(\beta^{-3}) \tag{46}$$

$$\Delta_e = \chi^{\frac{1}{4}} - \left[\frac{36\Gamma(\frac{3}{4})}{13\Gamma(\frac{1}{12})\Gamma(\frac{2}{3})}\right]\frac{\chi^{\frac{1}{3}}}{\beta} + \frac{4320}{91}\left[\frac{\Gamma(\frac{3}{4})\Gamma(\frac{5}{6})}{\Gamma^2(\frac{2}{3})\Gamma(\frac{1}{12})\Gamma(\frac{1}{6})}\right]\frac{\chi^{\frac{5}{12}}}{\beta^2} + \mathcal{O}(\beta^{-3}) \tag{47}$$

The leading terms in the preceding equations coincide with the case $\alpha \to \infty$, indicating that for this limit there is no influence of the longitudinal heat conduction of the plate on the film condensation process.

Otherwise, for small values of β, the integral term in Eqs. (44) reduces in a first approximation to

$$\int_0^{\sigma} K(\sigma, \sigma')\frac{\mathrm{d}\theta'_{we}}{\mathrm{d}\sigma'}\,\mathrm{d}\sigma' \sim -1 \qquad \text{for} \qquad \sigma \gg 1 \tag{48}$$

Therefore, the asymptotic solutions of Eqs. (44) for $\sigma \gg 1$ are given by

$$\psi \sim \left(\tfrac{9}{8}\right)^{\frac{4}{3}}\sigma^{\frac{8}{9}}, \qquad \theta_{we} \sim \left(\tfrac{9}{8}\right)^{\frac{1}{3}}\sigma^{-\frac{1}{9}} \tag{49}$$

Finally, the nondimensional condensed-layer thickness and temperature of the plate at the bottom end for $\beta \to 0$ are given by

$$\Delta_{eL} = \Delta_e(\chi = 1) \sim \left(\tfrac{9}{8}\right)^{\frac{1}{3}}\beta^{\frac{1}{3}}$$
$$\theta_{weL} = \theta_{we}(\chi = 1) \sim \left(\tfrac{9}{8}\right)^{\frac{1}{3}}\beta^{\frac{4}{3}} \tag{50}$$

V. Results

In the limit of very large values of β, the temperature of the plate is practically uniform and close to the value of the temperature of the cooling flow. This is the well-known Nusselt solution. Here, the longitudinal heat conduction does not play an important role, even for large values of α. On the other hand, for finite values of β and $\alpha \to 0$, the system of Eqs. (26) and (27) is singular. This means that it is necessary to include the inner heat conducting layers at both ends of the plate. However, these thermal conduction boundary layers have only a local influence, and we do not present the solution in these layers because it will unnecessarily increase the length of the paper. Outside these inner regions, there is an outer zone, where the longitudinal heat conduction through the plate is negligible in a first approximation. In this outer zone and for $\alpha = 0$, the nondimensional temperature of the heat conducting plate and the thickness of the condensate film (at $\chi = 1$) as a function of the parameter β are shown in Figs. 2 and 3, using asymptotic and numerical techniques. It is clear from Fig. 2 that, for increasing values of the parameter β, the temperature of the plate decreases because the thermal resistance of the condensed fluid becomes larger than the corresponding resistance of the cooling flow. In Fig. 3, the nondimensional thickness of the film is shown to increase for increasing values of the parameter β. The asymptotic solutions in the limit $\alpha = 0$ for $\beta \to 0$ and $\beta \to \infty$ are also plotted, together with the leading term for $\alpha \to \infty$, given by Eqs. (40) and (34). The influence of the longitudinal heat conduction through the plate is very small, as we can see from Fig. 3. However, we obtain slightly more condensation at the lower end of the plate as the value of α decreases, as predicted by the first-order correction given by Eq. (38), where $\Delta_1(1)$ is positive for all values of β. For large values of β, the asymptotic solutions give

$$\Delta(1) \sim 1 - \frac{0.22222}{\beta} + \frac{0.2963}{\beta^2} + \cdots \qquad \text{for} \qquad \alpha \to \infty \tag{51}$$

$$\Delta(1) = 1 - \frac{0.21793}{\beta} + \frac{0.55947}{\beta^2} + \cdots \qquad \text{for} \qquad \alpha = 0 \tag{52}$$

Figure 4 shows the numerical results of the nondimensional temperature of the plate as a function of χ for $\alpha = 0$ and different values of β.

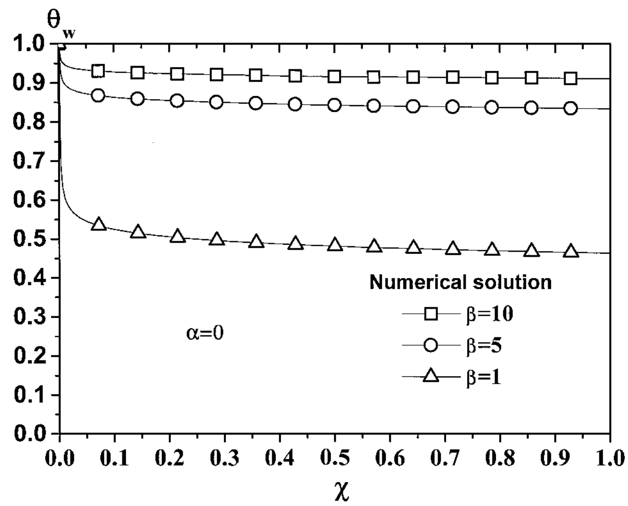

Fig. 4 Nondimensional plate temperature as a function of χ for different values of β, where $\alpha = 0$.

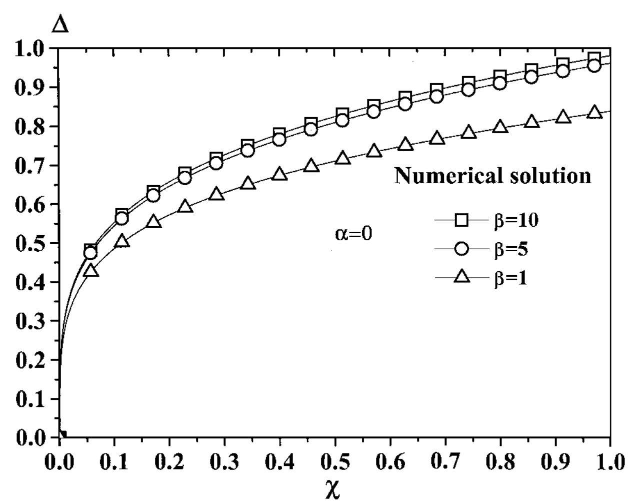

Fig. 5 Nondimensional condensed-layer thickness as a function of χ for different values of β, where $\alpha = 0$.

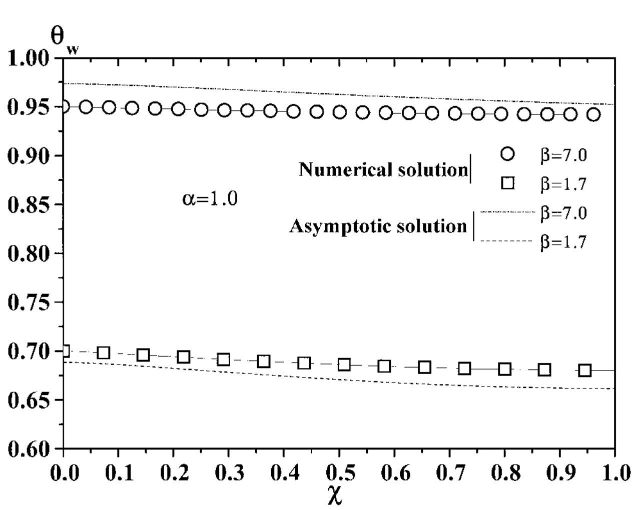

Fig. 6 Nondimensional plate temperature as a function of χ for different values of β, where $\alpha = 1$.

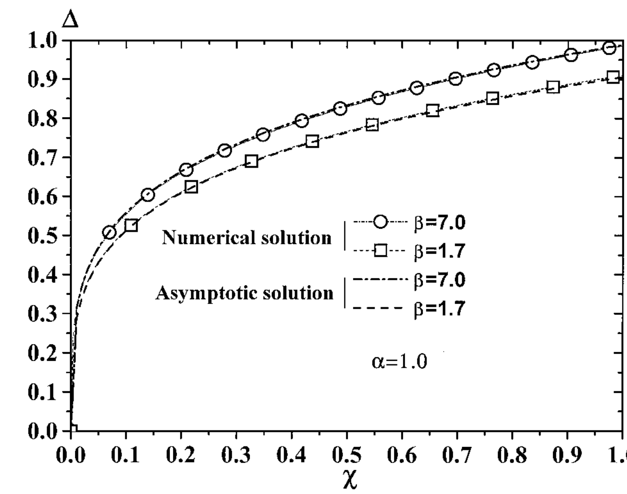

Fig. 7 Nondimensional condensed-layer thickness as a function of χ for different values of β, where $\alpha = 1$.

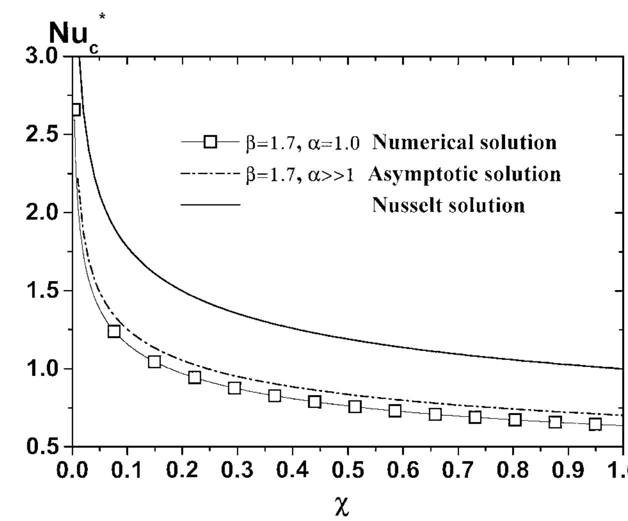

Fig. 8 Reduced Nusselt number for the condensed fluid for $\alpha \gg 1$ and $\alpha = 1$ with $\beta = 1.7$; the Nusselt solution is also presented.

The nondimensional temperature of the plate always decreases along the plate. However, this effect is even more important for small values of β. The results also give the Nusselt classical solution as a particular case. For $\alpha = 0$, Fig. 5 shows the nondimensional condensed-layer thickness as a function of χ for different values of β. We can see that the nondimensional thickness of the condensed fluid always increases with increasing values of β, and this asymptotically attains the Nusselt solution for $\beta \to \infty$.

For $\alpha \sim 1$ and finite values of β, the problem defined by Eqs. (26) and (27) can be solved numerically. Figure 6 shows the corresponding numerical and asymptotic solutions for the nondimensional temperature as a function of the longitudinal coordinate χ for different values of β and for $\alpha = 1$. The temperature profiles in the plate tend to be uniform. The asymptotic solution for large values of α introduces a 3% error for values of α around unity. Similarly, Fig. 7 shows the results for the nondimensional thickness of the condensed layer as a function of χ for different values of β and for $\alpha = 1$. Figures 6 and 7 show the great influence of parameter β. For comparison, the two-term asymptotic solution obtained for large values of α gives a reasonable agreement even for values of α of order unity. Finally, Fig. 8 shows the reduced local Nusselt number for the condensed fluid, indicating the influence of the variable temperature through the plate.

VI. Conclusion

The laminar film condensation process of a saturated vapor in contact with the external surfaces of two thin vertical heat conducting plates has been analyzed for small values of Jakob numbers, using asymptotic as well as numerical techniques. The finite thermal conductivity of the plate material allows heat transfer by conduction upstream through the plate. The local-heat-forced convection through the inner lateral surfaces of the plates, strongly affected by the axial heat conduction, governs the spatial evolution of the plate temperature and the condensed-layer thickness. Although there is a great influence on the temperature distribution by the longitudinal conduction effects through the plates, there is only a slight influence on the mass flow rate of condensate at the lower edge of the plate. Decreasing the longitudinal heat conductance α produces a slight increase in the global condensation rate. This is true for the thermally thin wall regime, that is, for values of α very large compared with ε^2. In the thermally thick wall regime ($\alpha/\varepsilon^2 \sim 1$), it is inverted, producing a decreasing condensation rate for smaller values of the parameter α. This regime is not considered here.

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